Gravitational clustering in Static and Expanding Backgrounds

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A brief summary of several topics in the study of gravitational many body problem is given. The discussion covers both static backgrounds (applicable to astrophysical systems) as well as clustering in an expanding background (relevant for cosmology).

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INTRODUCTION

The statistical mechanics of systems dominated by gravity is of interest both from the theoretical and "practical" perspectives. Theoretically, this field has close connections with areas of condensed matter physics, fluid mechanics, renormalization group, etc. From the practical point of view, the ideas find application in different areas of astrophysics and cosmology, especially in the study of globular clusters, galaxies and gravitational clustering in the expanding universe. [For a review of statistical mechanics of gravitating systems in static background, see [1]; textbook descriptions are in [2]; gravitational clustering in cosmology is reviewed in [3] and in the textbooks [4]; for a sample of different approaches see [5] and the references cited therein.]

GRAVITATIONAL CLUSTERING IN STATIC BACKGROUND

To construct the statistical description of a system of N self gravitating point particles, one should begin with the construction of the micro canonical ensemble describing such a system. If $g(E)$ is the volume of the constant energy surface $H(p_i, q_i) = E$, then the entropy and the temperature of the system will be $S(E) = \ln g(E)$ and $T(E) \equiv \beta(E)^{-1} = (\partial S/\partial E)^{-1}$. (The finiteness of g requires the system to be confined to a finite volume in space for *any* system).

Systems for which a description based on canonical ensemble is possible, the Laplace transform of $g(E)$ with respect to a variable β will give the partition function $Z(\beta)$. Gravitating systems of interest in astrophysics, however, cannot be described by a canonical ensemble [1], [6]. Virial theorem holds for such systems and we have $(2K + U) = 0$, where K and U are the total kinetic and potential energies of the system. This leads to $E = K+U = -K$; since the temperature of the system is proportional to the total kinetic energy, the specific heat will be negative: $C_V \equiv (\partial E/\partial T)_V \propto (\partial E/\partial K) < 0$. On the other hand, the specific heat of any system described by a canonical ensemble $C_V = \beta^2\langle(\Delta E)^2\rangle$ will be positive definite. Thus, one cannot describe self gravitating systems of the kind we are interested in by canonical ensemble.

One can, however, attempt to find the equilibrium configuration for self gravitating systems by maximising the entropy $S(E)$ or the phase volume $g(E)$. For a system of point particles, there is again no global maximum for entropy [1],[2]. If we move a small number of these particles arbitrarily close to each other, the potential energy of interaction of a pair of these particles, $-Gm_1m_2/r_{12}$, will become arbitrarily high as $r_{12} \to 0$. Transferring some of this energy to the rest of the particles, we can increase their kinetic energy without limit. This will clearly increase the phase volume occupied by the system (in the momentum space) without bound. This argument can be made more formal by dividing the original system into a small, compact core, and a large diffuse halo and allowing the core to collapse and transfer the energy to the halo.

The absence of the global maximum for entropy - as argued above - depends on the lack of small scale cutoff. If we assume, instead, that each particle has a radius a, there will be an upper bound on the amount of energy that can be made available to the rest of the system. Further, no real system is completely isolated and to obtain a truly isolated system, we need to confine the system inside a spherical region of radius R with, say, reflecting wall.

The two cut-offs a and R will make the upper bound on the entropy finite, but even with the two cut-offs the formation of a compact core and a diffuse halo will still occur, since this is the direction of increasing entropy. Particles in the hot diffuse component will permeate the entire spherical cavity, bouncing off the walls and having a kinetic energy which is significantly larger than the potential energy. The compact core will exist as a gravitationally bound system with very little kinetic energy. A formal way of understanding this phenomena is based on the virial theorem [2]:

$$2T + U = 3PV + \Phi \tag{1}$$

for a system with a short distance cut-off confined to a sphere of volume V, where P is the pressure on the walls and Φ is the correction to the potential energy arising from the short distance cut-off. This equation can be satisfied in essentially three different ways. If T and U are significantly higher than $3PV$ and Φ, then we have $2T + U \approx 0$ which describes a self gravitating systems

in standard virial equilibrium, but not in the state of maximum entropy. If $T \gg U$ and $3PV \gg \Phi$, one can have $2T \approx 3PV$ which describes an ideal gas with no potential energy confined to a container of volume V; this will describe the hot diffuse component at late times. If $T \ll U$ and $3PV \ll \Phi$, then one can have $U \approx \Phi$, describing the compact potential energy dominated core at late times. Such an asymptotic state with two distinct phases is quite different from what would have been expected for systems with only short range interaction. If the gravitating system is put in a heat bath and the temperature is varied, a sudden phase transition occurs at a critical temperature, leading to the formation of the two phases [7], [1].

There are, however, configurations which are *local extrema* of entropy, which are not global maxima. Intuitively, one would have expected the distribution of matter in such configuration to be described by a Boltzmann distribution, with the $\rho(\mathbf{x}) \propto \exp[-\beta\phi(\mathbf{x})]$, where ϕ is the gravitational potential related to the density ρ by Poisson equation. This configuration, called isothermal sphere, has a density profile $\rho \propto x^{-2}$ asymptotically. Isothermal spheres with total energy E and mass M, however, cannot exist [8] if $(RE/GM^2) < -0.335$. Even when $(RE/GM^2) > -0.335$, the isothermal solution need not be stable. The stability of this solution can be investigated by studying the second variation of the entropy. Such a detailed analysis shows that the following results are true: (i) Systems with $(RE/GM^2) < -0.335$ cannot evolve into isothermal spheres. Entropy has no extremum for such systems [1], [8]. (ii) Systems with $((RE/GM^2) > -0.335)$ and $(\rho(0) > 709\,\rho(R))$ can exist in a meta-stable (saddle point state) isothermal sphere configuration. Here $\rho(0)$ and $\rho(R)$ denote the densities at the center and edge respectively. The entropy extrema exist but they are not local maxima. (iii) Systems with $((RE/GM^2) > -0.335)$ and $(\rho(0) < 709\,\rho(R))$ can form isothermal spheres, which are local maximum of entropy. These are striking peculiarities in the case of SMGS and seem to find application in the physics of globular clusters.

GRAVITATIONAL CLUSTERING IN AN EXPANDING BACKGROUND

There is considerable amount of observational evidence to suggest that one of the dominant energy densities in the universe is contributed by self gravitating (nearly) point particles. The smooth average energy density of these particles drive the expansion of the universe while any small deviation from the homogeneous energy density will cluster gravitationally. It is often enough (and necessary) to use a statistical description and relate different statistical indicators (like the power spectra, nth order correlation functions, ...) of the resulting density distribution to the statistical parameters (usually the power spectrum) of the initial distribution.

The relevant scales at which gravitational clustering is nonlinear are less than about 10 Mpc, while the expansion of the universe has a characteristic scale of about 4000 Mpc [4]. Hence, nonlinear gravitational clustering in an expanding universe can be adequately described by Newtonian gravity by introducing a *proper* coordinate for the $i-$th particle $\mathbf{r}_i$, related to the *comoving* coordinate $\mathbf{x}_i$, by $\mathbf{r}_i = a(t)\mathbf{x}_i$ where $a(t)$ is the expansion factor. The Newtonian dynamics works with the proper coordinates $\mathbf{r}_i$ which can be translated to the behaviour of the comoving coordinate $\mathbf{x}_i$ by this rescaling.

If $\mathbf{x}(t, \mathbf{q})$ is the position of a particle at time t with its initial position being $\mathbf{q}$, then equations for gravitational clustering in an expanding universe, in the Newtonian limit, can be summarised as [10], [3].

$$\ddot{\mathbf{x}} + \frac{2\dot{a}}{a}\dot{\mathbf{x}} = -\frac{1}{a^2}\nabla_{\mathbf{x}}\phi; \tag{2}$$

$$\begin{aligned}\ddot{\phi}_{\mathbf{k}} + 4\frac{\dot{a}}{a}\dot{\phi}_{\mathbf{k}} &= -\frac{1}{2a^2}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\phi_{\frac{\mathbf{k}}{2}+\mathbf{p}}\phi_{\frac{\mathbf{k}}{2}-\mathbf{p}}\mathcal{G}(\mathbf{k},\mathbf{p}) \\ &+ \left(\frac{3H_0^2}{2}\right)\int\frac{d^3\mathbf{q}}{a}\left(\frac{\mathbf{k}.\dot{\mathbf{x}}}{k}\right)^2 e^{i\mathbf{k}.\mathbf{x}}, \end{aligned} \tag{3}$$

where $\mathbf{x} = \mathbf{x}(t, \mathbf{q}), \mathcal{G}(\mathbf{k}, \mathbf{p}) = (k/2)^2 + p^2 - 2(\mathbf{k}.\mathbf{p}/k)^2$ and $\phi_{\mathbf{k}}(t)$ is the Fourier transform of the gravitational potential $\phi(t, \mathbf{x})$ due to perturbed density.

Equation (3) is exact but involves $\dot{\mathbf{x}}(t, \mathbf{q})$ on the right hand side and hence, cannot be considered as closed. Together, the two equations form a closed set but solving them exactly is an impossible task. It is, however, possible to use this equation with several well motivated approximations [9] to obtain information about the system. I shall briefly mention a few of them.

Consider first the effect of a bunch of particles, in a virialized cluster, on the rest of the system. This is described, to the lowest order, by just the monopole moment of the cluster – which can be taken into account by replacing the cluster by a single particle at the centre of mass having appropriate mass. Such a replacement should not affect the evolution at scales much bigger than the cluster size. At first sight, one may wonder how this feature ("renormalizability of gravity") is taken care of in equation (3). Inside a galaxy cluster, for example, the velocities $\dot{\mathbf{x}}$ can be quite high and one might wonder whether this could influence the evolution of $\phi_{\mathbf{k}}$ at all scales. This does not happen and, to the lowest order, the contribution from virialized bound clusters cancel [9], [10] in the two terms in the right hand side of (3).

In this limit, $\phi_{\mathbf{k}}$ is constant in time and the Poisson equation $-k^2\phi_{\mathbf{k}} = 4\pi G\rho_b a^2\delta_{\mathbf{k}} \propto (\delta_{\mathbf{k}}/a)$ in the matter dominated universe with $a(t) \propto t^{2/3}, \rho_b \propto a^{-3}$

implies that the density contrast has the growing solution $\delta_{\mathbf{k}}(t) = [a(t)/a(t_i)]\delta_{\mathbf{k}}(t_i)$. The power spectrum $P(\mathbf{k},t) = \langle|\delta_{\mathbf{k}}(t)|^2\rangle$ and the correlation function $\xi(\mathbf{x},t)$ [which is the Fourier transform of the power spectrum] both grow as $a^2(t)$. This allows us to fix the evolution of clustering at sufficiently large scales uniquely. The clustering at these scales, which is well described by linear theory, grows as a^2.

There is, however, an important caveat to this claim. While ignoring the right hand side of (3) one is comparing its contribution at any wave number $\mathbf{k}$ to the contribution in linear theory. If at the relevant wavenumber, the contribution from linear evolution is negligibly small, then the *only* contribution will come from the terms on the right hand side and, of course, we cannot ignore it in this case. This contribution will scale as k^2R^2, where R is the typical scale of virialized systems and will lead to a development of $\delta_{\mathbf{k}} \propto k^2, P(k) \propto k^4$ at small k. Thus, if the large scales have too little power intrinsically (i.e., if $n > 4$), then the long wavelength power will soon be dominated by the "k^4 - tail" of the short wavelength power arising from the nonlinear clustering. This is an interesting and curious result which is characteristic of gravitational clustering.

NONLINEAR SCALING RELATIONS

As to be expected, cosmological expansion completely changes the asymptotic nature of the problem. The problem has now become time dependent and it will be pointless to look for "equilibrium solutions".

There are three key theoretical questions which are of considerable interest in this area which I will briefly summarise:

- If the initial power spectrum is sharply peaked in a narrow band of wavelengths, how does the evolution transfer the power to other scales? (This is, in some sense, analogous to determining the Green function for the gravitational clustering except that superposition will not work in the nonlinear context.)
- Do the virialized structures formed in an expanding universe due to gravitational clustering have any invariant properties? Can their structure be understood from first principles?
- Does the gravitational clustering at late stages wipe out the memory of initial conditions or does the late stage evolution depend on the initial power spectrum of fluctuations?

To make any progress with these questions we need a robust prescription which will relate statistical indicators like the two-point correlation function in the nonlinear regime to the initial power spectrum. Fortunately, this problem has been solved [11] to a large extent and hence, one can use this as a basis for attacking these questions.

The nonlinear mean correlation function can be expressed in terms of the linear mean correlation function by the relation:

$$\bar{\xi}(a,x) = \begin{cases} \bar{\xi}_L(a,l) & (\text{for } \bar{\xi} < 1) \\ \bar{\xi}_L(a,l)^D & (\text{for } 1 < \bar{\xi} < 125) \\ 11.7\bar{\xi}_L(a,l)^{Dh/2} & (\text{for } 125 < \bar{\xi}) \end{cases} \quad (4)$$

where $l = x[1+\bar{\xi}(a,x)]^{1/D}$, $D = 2,3$ is the dimension of space and h is a constant. [The results of numerical simulation in 2D, suggests that $h = 3/4$ asymptotically. We will discuss the 3D results in more detail below]. The numerical values are for $D = 3$.

One could use this to examine whether the power spectrum (or the correlation function) has a universal shape at late times, independent of initial power spectrum. This is indeed true [12] if the initial spectrum was sharply peaked. In this case, at length scales smaller than the initial scale at which the power is injected, the two point correlation function has a universal asymptotic shape of $\bar{\xi}(a,x) \propto a^2x^{-1}(L+x)^{-1}$, where L is the length scale at which $\bar{\xi} \approx 200$. This can be understood as follows:

In the quasi-linear phase, regions of high density contrast will undergo collapse and in the nonlinear phase more and more virialized systems will get formed. We recall that, in the study of finite gravitating systems made of point particles and interacting via Newtonian gravity, isothermal spheres play an important role and are the local maxima of entropy. Hence, dynamical evolution drives the system towards an $(1/x^2)$ profile. Since, one expects similar considerations to hold at small scales, during the late stages of evolution of the universe, we may hope that isothermal spheres with $(1/x^2)$ profile may still play a role in the late stages of evolution of clustering in an expanding background. However, while converting the density profile to correlation function, we need to distinguish between two cases. In the quasi-linear regime, dominated by the collapse of high density peaks, the density profile around any peak will scale as the correlation function and we will have $\bar{\xi} \propto (1/x^2)$. On the other hand, in the nonlinear end, we will be probing the structure inside a single halo and $\xi(\mathbf{x})$ will vary as $\langle\rho(\mathbf{x}+\mathbf{y})\rho(\mathbf{y})\rangle$. If $\rho \propto |x|^{-\epsilon}$, then $\bar{\xi} \propto |x|^{-\gamma}$ with $\gamma = 2\epsilon - 3$. This gives $\bar{\xi} \propto (1/x)$ for $\epsilon = 2$. Thus, if isothermal spheres are the generic contributors, then we expect the correlation function to vary as $(1/x)$ and nonlinear scales, steepening to $(1/x^2)$ at intermediate scales. Further, since isothermal spheres are local maxima of entropy, a configuration like this should remain undistorted for a long duration. This argument suggests that a $\bar{\xi}$ which goes as $(1/x)$ at small scales and $(1/x^2)$ at intermediate scales is likely to grow approximately as a^2 at all scales. At scales bigger than the scale at which power was originally injected, the

spectrum develops a k^4 tail for reasons described before. This is confirmed by simulations for sharply peaked initial spectra. But if the initial spectrum is *not* sharply peaked, each band of power evolves by this rule and the final result is a lot messier.

The second question one could ask, concerns the density profiles of individual virialized halos. If the density field $\rho(a,\mathbf{x})$ at late stages can be expressed as a superposition of several halos, each with some density profile $f(\mathbf{x})$ then the i-th halo centred at $\mathbf{x}_i$ will contribute a density $f(\mathbf{x}-\mathbf{x}_i,a)$ at the location $\mathbf{x}$. The power spectrum for the density contrast, $\delta(a,\mathbf{x}) = (\rho/\rho_b - 1)$, will be $P(k) = |f(k)|^2 P_c(k)$, where $P_c(\mathbf{k},a)$ denotes the power spectrum of the distribution of centers of the halos. If the correlation function $\bar{\xi} \propto x^{-\gamma}$, the correlation function of the centres $\bar{\xi} \propto x^{-\gamma_c}$ and the individual profiles are of the form $f(x) \propto x^{-\epsilon}$, then this relation translates to $\epsilon = 3 + (1/2)(\gamma - \gamma_c)$.

At very nonlinear scales, the centres of the virialized clusters will coincide with the deep minima of the gravitational potential. Hence, the power spectrum of the centres will be proportional to the power spectrum of the gravitational potential $P_\phi(k) \propto k^{n-4}$ if $P(k) \propto k^n$. Since the correlation functions vary as $x^{-(\alpha+3)}$ when the power spectra vary as k^α, it follows that $\gamma = \gamma_c - 4$. Substituting into the above relation, we find that $\epsilon = 1$ at the extreme nonlinear scales. On the other hand, in the quasi-linear regime, reasonably large density regions will act as cluster centres and hence, one would expect $P_c(k)$ and $P(k)$ to scale in a similar fashion. This leads to $\gamma \approx \gamma_c$, giving $\epsilon \approx 3$. So we would expect the halo profile to vary as x^{-1} at small scales steepening to x^{-3} at large scales. A simple interpolation for such a density profile will be

$$f(x) \propto \frac{1}{x(x+l)^2}. \qquad (5)$$

Such a profile, usually called NFW profile [13], is often used in cosmology. The argument given above, however, is very tentative and it is difficult to obtain (5) from a more rigorous theoretical analysis.

In fact, it is possible to reach different conclusions regarding the asymptotic evolution of the system from different physical assumptions [14]. The NSR in (4) for 3-D with constant h leads to the asymptotic correlation function

$$\bar{\xi}(a,x) \propto a^{\frac{2\gamma}{n+3}} x^{-\gamma}; \qquad \gamma = \frac{3h(n+3)}{2+h(n+3)} \qquad (6)$$

for an initial spectrum which is scale-free power law with index n. If we assume that the evolution gets frozen in proper coordinates at highly nonlinear scales then it is easy to show that $h=1$. *If* this assumption (called stable clustering) *is* valid, then the late time behaviour of $\bar{\xi}(a,x)$ is strongly dependent on the initial conditions and (6) shows that $\bar{\xi}(a,x)$ at nonlinear scales will be as,

$$\bar{\xi}(a,x) \propto a^{\frac{6}{n+5}} x^{-\frac{3(n+3)}{n+5}}; \qquad (\bar{\xi} \gg 200). \qquad (7)$$

In other words the two (apparently reasonable) requirements: (i) validity of stable clustering at highly nonlinear scales and (ii) the independence of late time behaviour from initial conditions, *are mutually exclusive*. [At present, there exists some evidence from simulations [15] that this process, called stable clustering, does *not* occur in the $a \propto t^{2/3}$ cosmological model; but this result is not definitive].

In the very nonlinear limit, the correlation function probes the interiors of individual halos and we have $\epsilon = (1/2)(3+\gamma)$. [This corresponds to $P_c =$ constant, $\gamma_C = 3$ in the previous discussion.] If γ depends on n so will ϵ and the individual halos will remember the initial power spectrum.

We can obtain a γ which is independent of initial power law index provided h satisfies the condition $h(n+3) = c$, a constant. In this case, the halo profile will be given by $\epsilon = 3(c+1)/(c+2)$. Note that we are now demanding the asymptotic value of h to *explicitly depend* on the initial conditions though the *spatial* dependence of $\bar{\xi}(a,x)$ does not. As an example of the power of such a — seemingly simple — analysis, note the following: Since $c \geq 0$, it follows that $\epsilon > (3/2)$; invariant profiles with shallower indices (for e.g with $\epsilon = 1$) discussed above are not consistent with the evolution described above. One requires very high resolution simulations to verify the condition $h(n+3) = c$ and the current results are inconclusive.

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